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Explicit generating functional for pions and virtual photons

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Abstract: We construct the explicit one-loop functional of chiral perturbation theory for two light flavours, including virtual photons. We stick to contributions where 1 or 2 mesons and at most one photon are running in the loops. With the explicit functional at hand, the evaluation of the relevant Green functions boils down to performing traces over the flavour matrices. For illustration, we work out the $\pi^+\pi^- \to \pi^0\pi^0$ scattering amplitude at threshold at order p^4, e^2p^2.

Keywords: QCD, Chiral Lagrangians.

Contents

1. Introduction

The low-energy structure of QCD may be investigated in the context of chiral perturbation theory [1, 2, 3]. In refs. [2, 3] Gasser and Leutwyler constructed the generating functional of $\mathrm{SU}(N_f)\times\mathrm{SU}(N_f)$ for $N_f = 2, 3$ to order p^4. An explicit representation of this functional, including graphs with up to two propagators and at most four external pion fields,[1] was also given. For the first two flavours u and d, this construction was extended recently to include graphs with three propagators [4].

[1]The formula presented also includes external vector, axial vector, scalar and pseudoscalar fields.

In the case of QCD including electromagnetic interactions, the initial theory depends on the strong coupling constant g, the fine structure constant $\alpha \simeq 1/137$ and the light quark masses. The corresponding effective theory was formulated in refs. [5]–[8]. It is based on a systematic expansion, which combines the chiral power counting scheme with the expansion in powers of the electromagnetic coupling e. Within this framework, virtual photon effects were calculated for a number of processes. In the two-flavour case, electromagnetic corrections to the $\pi - \pi$ scattering amplitude [7, 8, 9] as well as to the vector and scalar form factors [10] have been evaluated at next-to-leading order. Further, pionic beta decay and radiative τ decay have been analyzed [11, 12] in a generalized framework including leptons and virtual photons [13]. Virtual photons have also been included in the three-flavour case [14]–[18].

The purpose of the present article is to include virtual photons in the explicit generating functional at one-loop [2]. The advantage of having the explicit functional at hand is evident: The calculation of S-matrix elements boils down to performing traces over flavour matrices - the combinatorics has already been carried out and all quantities are expressed in terms of ultraviolet finite integrals. In the following, we stick to the two-flavour case. The extension to three flavours will be considered elsewhere [19].

The article is organized as follows: In the first part (sections 2, 3), we construct the generating functional for $\mathrm{SU(2)_R \times SU(2)_L \times U(1)_V}$ to $\mathcal{O}(p^4, e^2p^2)$ in the low-energy expansion. This allows us to calculate Green functions to next-to-leading order in a simultaneous expansion in powers of the external momenta, of the quark masses and of the electromagnetic coupling. In order to extract form factors or scattering amplitudes from our explicit representation of the generating functional, one simply has to perform traces over flavour matrices. The extraction procedure is demonstrated in sections 5 and 6 by means of the $\pi^+\pi^- \to \pi^0\pi^0$ scattering amplitude.

2. Lagrangian

The variables of the effective theory are the pion field $\mathrm{U}(x) \in \mathrm{SU(2)}$ and the photon field $A_\mu(x)$. As shown in ref. [6], the electric charge must be - for consistency - treated as a quantity of order p in the chiral expansion.

2.1 Leading order

At $\alpha \neq 0$, the leading order lagrangian which is consistent with chiral symmetry becomes [5, 6],

$$\begin{aligned}\mathcal{L}^{(2)} = & \frac{F^2}{4}\langle d^\mu U^+ d_\mu U + \chi^+ U + U^+\chi\rangle - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \\ & - \frac{1}{2a}(\partial^\mu A_\mu)^2 + C\langle Q_\mathrm{R} U Q_\mathrm{L} U^+\rangle \,,\end{aligned} \tag{2.1}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \qquad d_\mu U = \partial_\mu U - iR_\mu U + iUL_\mu \,, \qquad \chi = 2B(s+ip) \tag{2.2}$$

and

$$
\begin{aligned}
R_\mu &= v_\mu + A_\mu Q_{\rm R} + a_\mu \,, \\
L_\mu &= v_\mu + A_\mu Q_{\rm L} - a_\mu \,.
\end{aligned}
\tag{2.3}
$$

The symbol $\langle \ldots \rangle$ denotes the trace in flavour space. The external fields v_μ, a_μ, p and s are given by

$$
\begin{aligned}
v_\mu &= \frac{1}{2} v^0 \mathbf{1} + v^i \frac{\tau^i}{2} \,, \qquad a_\mu = a^i \frac{\tau^i}{2} \,, \\
s &= s^0 \mathbf{1} + s^i \tau^i \,, \qquad p = p^0 \mathbf{1} + p^i \tau^i \,,
\end{aligned}
\tag{2.4}
$$

where τ^i denote the Pauli matrices. We restrict ourselves to isovector axial fields, i.e. we take $\langle a_\mu \rangle = 0$. For the transformation properties of the external fields, we refer to [2]. The mass matrix of the two light quarks is contained in s,

$$
s = \mathcal{M} + \cdots \,, \qquad \mathcal{M} = \mathrm{diag}\,(m_u, m_d) \,. \tag{2.5}
$$

To ensure the chiral symmetry of the effective lagrangian (2.1), the local right- and left-handed spurions $Q_{\rm R}$ and $Q_{\rm L}$ transform under $\mathrm{SU}(2)_{\rm R} \times \mathrm{SU}(2)_{\rm L}$, according to

$$
Q_I = V_I Q_I V_I^+ \,, \qquad I = \mathrm{R}, \mathrm{L} \,. \tag{2.6}
$$

In the following, we work with a constant charge matrix

$$
Q_{\rm R} = Q_{\rm L} = Q = \frac{e}{3} \mathrm{diag}\,(2, -1) \,. \tag{2.7}
$$

The quantity a denotes the gauge fixing parameter and the parameters F, B and C are the three low-energy coupling constants occurring at leading order. In the chiral limit and in absence of electromagnetic interactions, F coincides with the pion decay constant, normalized such that $F_\pi = 92.4$MeV [20], while B is related to the quark condensate. The coupling constant C occurs in the low-energy expansion of the charged physical pion mass only.

Finally, the matrix $\bar{U}$ is determined by the equation of motion

$$
d_\mu d^\mu \bar{U} \bar{U}^+ - \bar{U} d_\mu d^\mu \bar{U}^+ + \bar{U} \chi^+ - \chi \bar{U}^+ - \frac{1}{2} \langle \bar{U} \chi^+ - \chi \bar{U}^+ \rangle + \frac{4C}{F^2} \left(\bar{U} Q \bar{U}^+ Q - Q \bar{U} Q \bar{U}^+ \right) = 0 \,, \tag{2.8}
$$

while the field equations for the photon field $\bar{A}_\mu$ read

$$
\left[g_{\mu\nu} \Box - \left(1 - \frac{1}{a} \right) \partial_\mu \partial_\nu \right] \bar{A}^\nu + \frac{i F^2}{2} \langle d_\mu \bar{U} [\bar{U}^+, Q] \rangle = 0 \,. \tag{2.9}
$$

2.2 Next-to-leading order

The next-to-leading order lagrangian reads

$$
\mathcal{L}^{(4)} = \mathcal{L}_{p^4} + \mathcal{L}_{p^2 e^2} + \mathcal{L}_{e^4} \,. \tag{2.10}
$$

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The lagrangian at order p^4 was constructed in refs. [2, 7, 21],[2]

$$
\begin{aligned}
\mathcal{L}_{p^4} = {} & \frac{l_1}{4}\langle d^\mu U^+ d_\mu U\rangle^2 + \frac{l_2}{4}\langle d^\mu U^+ d^\nu U\rangle\langle d_\mu U^+ d_\nu U\rangle + \frac{l_3}{16}\langle \chi^+ U + U^+\chi\rangle^2 + \\
& + \frac{l_4}{4}\langle d^\mu U^+ d_\mu \chi + d^\mu \chi^+ d_\mu U\rangle + l_5\langle \hat{R}_{\mu\nu} U \hat{L}^{\mu\nu} U^+\rangle + \\
& + \frac{i l_6}{2}\langle \hat{R}_{\mu\nu} d^\mu U d^\nu U^+ + \hat{L}_{\mu\nu} d^\mu U^+ d^\nu U\rangle - \frac{l_7}{16}\langle \chi^+ U - U^+\chi\rangle^2 + \\
& + \frac{1}{4}(h_1 + h_3)\langle \chi^+\chi\rangle + \frac{1}{2}(h_1 - h_3)\mathrm{Re}(\det\chi) - \\
& - \frac{1}{2}(l_5 + 4h_2)\langle \hat{R}_{\mu\nu}\hat{R}^{\mu\nu} + \hat{L}_{\mu\nu}\hat{L}^{\mu\nu}\rangle + \\
& + \frac{h_4}{4}\langle R_{\mu\nu} + L_{\mu\nu}\rangle\langle R^{\mu\nu} + L^{\mu\nu}\rangle \,,
\end{aligned} \tag{2.11}
$$

with right- and left-handed fields strengths defined as

$$
\begin{aligned}
R_{\mu\nu} &= \partial_\mu R_\nu - \partial_\nu R_\mu - i\,[R_\mu, R_\nu]\,, \qquad \hat{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\langle R_{\mu\nu}\rangle\,, \\
L_{\mu\nu} &= \partial_\mu L_\nu - \partial_\nu L_\mu - i\,[L_\mu, L_\nu]\,, \qquad \hat{L}_{\mu\nu} = L_{\mu\nu} - \frac{1}{2}\langle L_{\mu\nu}\rangle\,.
\end{aligned} \tag{2.12}
$$

The most general list of counterterms occurring at order p^2e^2 was given in refs. [7, 8],

$$
\begin{aligned}
\mathcal{L}_{p^2e^2} = {} & F^2\big\{k_1\langle d^\mu U^+ d_\mu U\rangle\langle Q^2\rangle + k_2\langle d^\mu U^+ d_\mu U\rangle\langle QUQU^+\rangle + \\
& + k_3\big(\langle d^\mu U^+ QU\rangle\langle d_\mu U^+ QU\rangle + \langle d^\mu U Q U^+\rangle\langle d_\mu U Q U^+\rangle\big) + \\
& + k_4\langle d^\mu U^+ QU\rangle\langle d_\mu U Q U^+\rangle + k_5\langle \chi^+ U + U^+\chi\rangle\langle Q^2\rangle + \\
& + k_6\langle \chi^+ U + U^+\chi\rangle\langle QUQU^+\rangle + \\
& + k_7\langle(\chi U^+ + U\chi^+)Q + (\chi^+ U + U^+\chi)Q\rangle\langle Q\rangle + \\
& + k_8\langle(\chi U^+ - U\chi^+)QUQU^+ + (\chi^+ U - U^+\chi)QU^+QU\rangle + \\
& + k_9\langle d_\mu U^+[c_{\mathrm{R}}^\mu Q, Q]U + d_\mu U[c_{\mathrm{L}}^\mu Q, Q]U^+\rangle + \\
& + k_{10}\langle c_{\mathrm{R}}^\mu Q U c_{\mathrm{L}\mu} Q U^+\rangle + k_{11}\langle c_{\mathrm{R}}^\mu Q c_{\mathrm{R}\mu} Q + c_{\mathrm{L}}^\mu Q c_{\mathrm{L}\mu} Q\rangle\big\}\,,
\end{aligned} \tag{2.13}
$$

where

$$
c_{\mathrm{I}}^\mu Q = -i[I^\mu, Q]\,, \qquad I = R, L\,. \tag{2.14}
$$

In the following we consider the next-to-leading order contributions where at most one virtual photon is running in a loop. Therefore, we drop the term $\mathcal{L}_{e^4}$ in eq. (2.10), (terms with k_{12}, k_{13} and k_{14} in ref. [7]).

The renormalized couplings are defined by

$$
\begin{aligned}
l_i &= l_i^r + \gamma_i\lambda\,, \\
h_i &= h_i^r + \delta_i\lambda\,, \\
k_i &= k_i^r + \sigma_i\lambda\,,
\end{aligned} \tag{2.15}
$$

[2]ref. [7] uses a different convention for the coupling constant h_2.

with

$$\lambda = \frac{\mu^{d-4}}{16\pi^2}\left[\frac{1}{d-4} - \frac{1}{2}\left(\ln 4\pi + \Gamma'(1) + 1\right)\right]. \tag{2.16}$$

The coupling constant h_4 is finite at $d = 4$ [21], because the singlet fields $\langle v_\mu \rangle$ and $A_\mu \langle Q \rangle$ do not occur in the leading order lagrangian (2.1). The coefficients γ_i, δ_i and σ_i are specified in refs. [2] and [7].

3. Generating functional at one-loop

To evaluate the one-loop contribution to the generating functional, we expand U and A_μ around their solutions $\bar{U} = u^2$ and $\bar{A}_\mu$ to the classical equations of motion,

$$\begin{aligned} U &= u\Big(1 + i\xi - \frac{1}{2}\xi^2 + \cdots\Big)u\,, \qquad \xi = \frac{1}{F}\xi^i\tau^i\,, \\ A_\mu &= \bar{A}_\mu + \epsilon_\mu\,. \end{aligned} \tag{3.1}$$

Collecting the fluctuations in $\eta = (\xi^1, \ldots, \xi^3, \epsilon^1, \ldots, \epsilon^4)$, the euclidean action can be written as a quadratic form [6, 7],

$$\int dx_E \mathcal{L}_E^{(2)} = \int dx_E \bar{\mathcal{L}}_E^{(2)} + \frac{1}{2}\int dx_E \eta^A D_E^{AB} \eta^B\,. \tag{3.2}$$

Here, $\bar{\mathcal{L}}_E^{(2)}$ is evaluated at the solution to the classical equations of motion. The one-loop functional $Z_{E\mathrm{oneloop}}$ then takes the form:

$$Z_{E\mathrm{oneloop}} = \frac{1}{2}\ln \det D_E\,. \tag{3.3}$$

3.1 Differential operator

In the Feynman gauge $a = 1$, the euclidean differential operator is given by

$$D_E^{AB} = -(\Sigma_\mu \Sigma_\mu)^{AB} + \Lambda^{AB} \qquad A, B = 1, \ldots, 7\,, \tag{3.4}$$

with

$$\Sigma_\mu = \partial_\mu \mathbf{1} + Y_\mu\,, \qquad Y_\mu = \begin{pmatrix} \Gamma_\mu^{ik} & X_\mu^{i\rho} \\ X_\mu^{\sigma k} & 0 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} \sigma^{ik} & \frac{1}{2}\gamma^{i\rho} \\ \frac{1}{2}\gamma^{\sigma k} & \rho\delta^{\sigma\rho} \end{pmatrix}. \tag{3.5}$$

The pion field indices i, k run from 1 to 3, while the photon field components are labeled by Greek letters $\sigma, \rho = 1, \ldots, 4$.

To renormalize the determinant we work in $d \neq 4$ dimensions,

$$\begin{aligned} \sigma^{ik} &= -\frac{1}{2}\langle [\Delta_\mu, \tau^i][\Delta_\mu, \tau^k]\rangle + \frac{1}{2}\delta^{ik}\langle\sigma\rangle - \frac{dF^2}{16}\langle H_-\tau^i\rangle\langle H_-\tau^k\rangle - \\ &\quad -\frac{C}{8F^2}\langle [H_+ + H_-, \tau^i][H_+ - H_-, \tau^k] + i \leftrightarrow k\rangle\,, \\ \Gamma_\mu^{ik} &= -\frac{1}{2}\langle [\tau^i, \tau^k]\Gamma_\mu\rangle\,, \\ \gamma_\mu^i &= F\langle ([H_+, \Delta_\mu] + \frac{1}{2}D_\mu H_-)\tau^i\rangle\,, \\ X_\mu^{i\rho} &= -X_\mu^{\rho i} = -\frac{F}{4}\langle H_-\tau^i\rangle\delta_\mu^\rho\,, \\ \rho &= \frac{3F^2}{8}\langle H_-^2\rangle\,, \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} D_\mu H_- &= \partial_\mu H_- + [\Gamma_\mu, H_-]\,, \\ \Gamma_\mu &= \frac{1}{2}\left[u^+, \partial_\mu u\right] - \frac{i}{2}u^+(v_\mu + \bar{A}_\mu Q_\mathrm{R} + a_\mu)u - \frac{i}{2}u(v_\mu + \bar{A}_\mu Q_\mathrm{L} - a_\mu)u^+\,, \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} \Delta_\mu &= \frac{1}{2}u^+ d_\mu \bar{U} u^+ = -\frac{1}{2}u d_\mu \bar{U}^+ u\,, \\ \sigma &= \frac{1}{2}\left(u^+\chi u^+ + u\chi^+ u\right)\,, \\ H_+ &= u^+ Q_\mathrm{R} u + u Q_\mathrm{L} u^+\,, \\ H_- &= u^+ Q_\mathrm{R} u - u Q_\mathrm{L} u^+\,. \end{aligned} \quad (3.8)$$

In the absence of external fields D_E reduces to

$$D_{0E} = \begin{pmatrix} (-\partial_\mu\partial_\mu + M_i^2)\delta^{ik} & \\ & -\partial_\mu\partial_\mu\delta^{\sigma\rho} \end{pmatrix}. \quad (3.9)$$

The parameters M_i denote the pion masses at leading order

$$M_1^2 = M_2^2 = M_+^2 = 2B\hat{m} + \frac{2Ce^2}{F^2}\,, \qquad M_3^2 = M_0^2 = 2B\hat{m}\,, \quad (3.10)$$

with $\hat{m} = (m_u + m_d)/2$. The full operator is given by

$$\begin{aligned} D_E &= D_{0E} + \delta_E\,, \\ \delta_E &= -\{\partial_\mu, Y_\mu\} - Y_\mu Y_\mu + \bar{\Lambda}\,, \end{aligned} \quad (3.11)$$

where

$$\bar{\Lambda} = \begin{pmatrix} \bar{\sigma}^{ik} & \frac{1}{2}\gamma^{i\rho} \\ \frac{1}{2}\gamma^{\sigma k} & \rho\delta^{\sigma\rho} \end{pmatrix}, \qquad \bar{\sigma}^{ik} = \sigma^{ik} - M_i^2\delta^{ik}\,. \quad (3.12)$$

3.2 Expansion in δ_E

To work out explicitly the one-loop contributions to a given Green function, we have to expand the determinant of the differential operator in powers of the external fields. Since δ_E vanishes if the external fields are switched off, we may expand $\ln\det D_E$ in powers of δ_E,

$$Z_{E\,\mathrm{oneloop}} = \frac{1}{2}\ln\det D_{0E} + \frac{1}{2}\langle D_{0E}^{-1}\delta_E\rangle - \frac{1}{4}\langle D_{0E}^{-1}\delta_E D_{0E}^{-1}\delta_E\rangle + \frac{1}{6}\langle D_{0E}^{-1}\delta_E D_{0E}^{-1}\delta_E D_{0E}^{-1}\delta_E\rangle + \cdots\,. \quad (3.13)$$

Following the counting scheme used in refs. [2, 3], the external fields a_μ and p count as $\mathcal{O}(\Phi)$, whereas v and $s-\mathcal{M}$ are of $\mathcal{O}(\Phi^2)$.

In the presence of electromagnetic interactions δ_E is of $\mathcal{O}(\Phi)$ rather than $\mathcal{O}(\Phi^2)$, since the quantities γ and X count as $\mathcal{O}(\Phi)$. To achieve an accuracy of $\mathcal{O}(\Phi^4)$, it is sufficient to stop the expansion in eq. (3.13) at $\mathcal{O}(\delta_E^3)$. This is due to the fact that both, γ and X, are of order e, and their contributions at $\mathcal{O}(\Phi^4)$ are beyond the precision of our calculation.

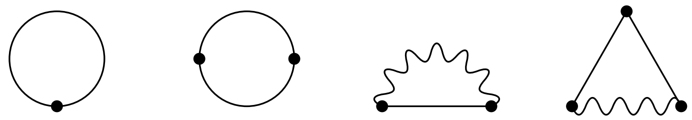

Figure 1: One-loop diagrams: The dots denote polynomials in the classical fields $\bar{U}$, $\bar{A}_\mu$, as well as in the external fields v_μ, a_μ, p and s. The plain line stands for the pion propagator and the wiggled line describes the photon.

3.3 Tadpole and unitarity contributions to order Φ^4

In the following, we work in Minkowski spacetime. If we add $\int dx(\bar{\mathcal{L}}^{(2)} + \bar{\mathcal{L}}^{(4)})$ to Z_{oneloop} and renormalize the low-energy couplings according to [2, 7], the result for the generating functional to $\mathcal{O}(p^4, e^2p^2)$ is ultraviolet finite as $d \to 4$. At order Φ^4, it takes the form, (see figure 1)

$$Z = Z_t + Z_u \,. \tag{3.14}$$

Explicitly, Z_t consists of

$$\begin{aligned} Z_t \;=\; & -\frac{1}{32\pi^2} M_i^2 \ln\frac{M_i^2}{\mu^2} \int dx\, \bar{\sigma}_{ii}(x) + \frac{1}{16\pi^2} M_i^2 \int dx Y_i(x) Y_i(x) + \\ & + \frac{1}{16\pi^2} \int dx\, \bar{\sigma}_{ik}(x) Y_k(x) Y_i(x) - \frac{1}{48\pi^2} \int dx D_\mu Y_i(x) D^\mu Y_i(x) + \\ & + \int dx\, \bar{\mathcal{L}}_2(x) + \int dx\, \bar{\mathcal{L}}_4^r(x) \,, \end{aligned} \tag{3.15}$$

where the first term contains tadpole contributions, the next three terms are generated by expanding Z_u around $d = 4$, while the last two terms are tree graphs. Repeated isospin indices are summed over. The quantity $\bar{\sigma}_{ik}$ now corresponds to[3]

$$\begin{aligned} \bar{\sigma}_{ik} \;=\; & \frac{1}{2} \langle [\Delta_\mu, \tau_i]\, [\Delta^\mu, \tau_k] \rangle + \frac{1}{2} \delta_{ik} \langle \sigma \rangle - \frac{dF^2}{16} \langle H_- \tau_i \rangle \langle H_- \tau_k \rangle - \\ & - \frac{C}{8F^2} \langle [H_+ {+} H_-, \tau_i]\, [H_+ {-} H_-, \tau_k] + i \leftrightarrow k \rangle - M_i^2 \delta_{ik} \,, \end{aligned} \tag{3.16}$$

and

$$Y_i = \frac{F}{4} \langle H_- \tau_i \rangle \,, \qquad D^\mu Y_i = \partial^\mu Y_i + \Gamma^\mu_{ik} Y_k \,. \tag{3.17}$$

The unitarity correction $Z_u = Z_{u_2} + Z_{u_3}$ contains one-loop graphs with two vertices,

$$\begin{aligned} Z_{u_2} \;=\; \int dx dy \Big\{ & M^r_{\mu\nu}(x{-}y, M_i^2, M_k^2) \Gamma^\mu_{ik}(x) \Gamma^\nu_{ki}(y) + \\ & + 2 M^{\mu r}_\mu (x{-}y, M_i^2, 0) Y_i(x) Y_i(y) + \\ & + K_\mu(x{-}y, M_i^2, M_k^2) \left[\Gamma^\mu_{ik}(x) \bar{\sigma}_{ki}(y) + d\Gamma^\mu_{ik}(x) Y_k(y) Y_i(y) \right] - \\ & - K_\mu(x{-}y, M_i^2, 0) \left[Y_i(x) \gamma_i^\mu(y) + 2 Y_i(x) \Gamma^\mu_{ik}(y) Y_k(y) \right] + \\ & + \frac{1}{4} J^r(x{-}y, M_i^2, M_k^2) \left[\bar{\sigma}_{ik}(x) \bar{\sigma}_{ki}(y) + 2d \bar{\sigma}_{ik}(x) Y_k(y) Y_i(y) \right] - \\ & - \frac{1}{8} J^r(x{-}y, M_i^2, 0) \left[\gamma_i^\mu(x) \gamma_{i\mu}(y) + 4 \gamma_{i\mu}(x) \Gamma^\mu_{ik}(y) Y_k(y) \right] \Big\} \,, \end{aligned} \tag{3.18}$$

[3]For further convenience, we specify $\bar{\sigma}_{ik}$ in $d \neq 4$ dimensions. However, eq. (3.15) is both, ultraviolet and infrared finite as $d \to 4$.

as well as the triangle vertex diagram with two pion and one photon propagator,

$$
\begin{aligned}
Z_{u_3} = \int dxdydz \Big\{ & \frac{1}{8} T\left(x-y, y-z, M_i^2, M_k^2\right) \gamma_i^\mu(x) \bar{\sigma}_{ik}(y) \gamma_{k\mu}(z) + \\
& + \frac{1}{4} T_{1\mu}\left(x-y, y-z, M_i^2, M_k^2\right) Y_i(x) \bar{\sigma}_{ik}(y) \gamma_k^\mu(z) - \\
& - \frac{1}{4} T_{2\mu}\left(x-y, y-z, M_i^2, M_k^2\right) \gamma_i^\mu(x) \bar{\sigma}_{ik}(y) Y_k(z) - \\
& - \frac{1}{8} T_{3\mu}\left(x-y, y-z, M_i^2, M_k^2\right) \gamma_i^\nu(x) \Gamma_{ik}^\mu(y) \gamma_{k\nu}(z) + \\
& + \frac{1}{4} T_{1\mu\nu}^r\left(x-y, y-z, M_i^2, M_k^2\right) Y_i(x) \Gamma_{ik}^\mu(y) \gamma_k^\nu(z) - \\
& - \frac{1}{4} T_{2\mu\nu}^r\left(x-y, y-z, M_i^2, M_k^2\right) \gamma_i^\mu(x) \Gamma_{ik}^\nu(y) Y_k(z) + \\
& + \frac{1}{2} T_{3\mu}^{\mu r}\left(x-y, y-z, M_i^2, M_k^2\right) Y_i(x) \bar{\sigma}_{ik}(y) Y_k(z) - \\
& - \frac{1}{2} T_{\mu\nu}^{\mu r}\left(x-y, y-z, M_i^2, M_k^2\right) Y_i(x) \Gamma_{ik}^\nu(y) Y_k(z) \Big\} . \qquad (3.19)
\end{aligned}
$$

For the definition of the various kernels, we refer the reader to appendix A. For non-exceptional momenta, the generating functional (3.15), (3.18) and (3.19) is finite as $d \to 4$. However, we want to extract on-shell matrix elements and this leads in general to infrared singularities. We treat these infrared pole terms using dimensional regularization. Both, the kernels as well as the flavour matrices are thus written in $d \neq 4$ dimensions.

In the isospin symmetry limit $m_u = m_d$ and $\alpha = 0$, our expression agrees with the explicit one-loop functional of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ in ref. [2].

The representation for the generating functional, specified in (3.15), (3.18) and (3.19), contains the one-loop contributions to Green functions in the even intrinsic parity sector, formed with the quark currents,

$$
S^i = \bar{q}\tau^i q\,, \qquad S^0 = \bar{q}q\,, \qquad P^i = \bar{q}i\gamma_5\tau^i q\,, \qquad P^0 = \bar{q}i\gamma_5 q\,, \qquad (3.20)
$$

and

$$
V_\mu^i = \frac{1}{2}\bar{q}\gamma_\mu\tau^i q\,, \qquad V_\mu^0 = \frac{1}{2}\bar{q}\gamma_\mu q\,, \qquad A_\mu^i = \frac{1}{2}\bar{q}\gamma_\mu\gamma_5\tau^i q\,. \qquad (3.21)
$$

The accuracy $\mathcal{O}(\Phi^4)$ suffices to calculate all two-point functions, the three-point functions containing one vector or scalar current, as well as the pseudoscalar and axial four-point functions.

The calculation of form factors or scattering amplitudes from the explicit representation of the generating functional (3.15), (3.18) and (3.19) amounts to perform the traces over the flavour matrices. In section 5, we demonstrate the extraction procedure by means of the $\pi - \pi$ scattering amplitude.

4. Masses and decay constants

In the following, it is convenient to work in σ-parameterization, i.e. to represent the matrix $\mathrm{U}(x)$ in the form

$$\mathrm{U}(x) = \sqrt{1 - \frac{\langle\phi^2\rangle}{2}} + i\phi\,, \qquad \phi = \frac{1}{F}\phi^i\tau^i\,, \tag{4.1}$$

however, all the steps listed are parameterization independent.

4.1 Generating functional at order Φ^2

We expand the generating functional in powers of Φ,

$$Z = \text{const} + Z^{(2)} + Z^{(4)} + \cdots\,, \tag{4.2}$$

where the pion field $\bar{\phi}$ counts as $\mathcal{O}(\Phi)$ and the photon field $\bar{A}_\mu$ is of order Φ^2. This can be seen by solving the equations of motion (2.8) and (2.9) to first and second order in Φ, respectively. The field equations for the pion field are discussed in subsection 5.1.

In the presence of electromagnetic interactions, the quadratic term $Z^{(2)}$ contains non-local contributions coming from Z_{u_2}. If we expand their Fourier transform around the physical pion masses M_{π_i}, we arrive at

$$\begin{aligned} Z^{(2)}\left(p, a_\mu, \bar{\phi}\right) = & \int dx - \frac{1}{2} Z_i^{-1}\bar{\phi}^i\left(\Box + M_{\pi_i}^2\right)\bar{\phi}^i + Z_i^{-\frac{1}{2}} F_{\pi_i}\partial^\mu a_\mu^i\bar{\phi}^i + \\ & + Z_i^{-\frac{1}{2}} G_{\pi_i} p^i\bar{\phi}^i + Z_3^{-\frac{1}{2}}\tilde{G}_\pi p^0\bar{\phi}^3(x) + \cdots\,. \end{aligned} \tag{4.3}$$

The ellipsis stands for terms which do not contribute to the residue of the pole in the axial or pseudoscalar two-point functions.

The scaling factors $Z_0 = Z_3$ and $Z_+ = Z_1 = Z_2$ are given by

$$\begin{aligned} Z_0 &= 1 - \frac{20e^2}{9}\left[k_1^r + k_2^r + \frac{9}{10}\left(k_4^r - 2k_3^r\right)\right] + \frac{M_+^2}{8\pi^2F^2}\ln\frac{M_+^2}{\mu^2}\,, \\ Z_+ &= 1 + 4e^2\lambda_{IR} - \frac{20e^2}{9}\left(k_1^r + k_2^r\right) + \frac{1}{16\pi^2F^2}\left[M_+^2\ln\frac{M_+^2}{\mu^2} + M_0^2\ln\frac{M_0^2}{\mu^2}\right]. \end{aligned} \tag{4.4}$$

We used dimensional regularization to treat the infrared singularity,

$$\lambda_{IR} = \frac{\mu^{d-4}}{16\pi^2}\left[\frac{1}{d-4} - \frac{1}{2}\left(\ln 4\pi + \Gamma'(1) + 1\right)\right]. \tag{4.5}$$

Note that the scaling factors Z_0 and Z_+ are parameterization independent, since unitarity determines the expansion of the field U to $\mathcal{O}(\Phi^2)$.

4.2 Physical masses

We choose the charged pion mass as normalization point [22]. In the isospin-symmetry limit $m_u = m_d$ and $\alpha = 0$, the position of the pole in the correlator of two axial currents is identified with the charged pion mass. The physical masses then read at next-to-leading order,[4]

$$\begin{aligned}
M_{\pi^+}^2 &= M_+^2 + M_{\pi^+}^2 \Bigg\{ \frac{e^2}{4\pi^2} - \frac{M_{\pi^+}^2 - 4e^2F^2Z}{32\pi^2F^2}\bar{l}_3 + \frac{e^2}{32\pi^2} \times \\
&\qquad \times \left[\left(3+\frac{4Z}{9}\right)\bar{k}_1 - \frac{40Z}{9}\bar{k}_2 - \frac{1}{9}(5+4Z)\bar{k}_5 + \frac{23}{9}(1+8Z)\bar{k}_6 + (1-8Z)\bar{k}_8 \right] + \\
&\qquad + \frac{4e^2}{9}k_7 - \frac{M_{\pi^+}^2 - 4e^2F^2Z}{32\pi^2F^2}\ln\frac{M_{\pi^+}^2}{M_{\pi^0}^2} \Bigg\} - \\
&\quad - \frac{4e^2B(m_d-m_u)}{3}k_7 + \mathcal{O}(e^4,p^6)
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
\Delta_\pi &= M_{\pi^+}^2 - M_{\pi^0}^2 = 2e^2F^2Z + \frac{M_{\pi^+}^2}{32\pi^2} \times \\
&\quad \times \left\{ e^2\left[8+3\bar{k}_3+4Z\bar{k}_4+2(1+8Z)\bar{k}_6+(1-8Z)\bar{k}_8\right] - \frac{2}{F^2}(M_{\pi^+}^2-4e^2F^2Z)\ln\frac{M_{\pi^+}^2}{M_{\pi^0}^2} \right\} + \\
&\quad + \frac{2B^2(m_d-m_u)^2}{F^2}l_7 + \mathcal{O}(e^4,p^6),
\end{aligned} \tag{4.7}$$

with $Z \doteq C/F^4$. The scale independent low-energy constants $\bar{l}_i$ and $\bar{k}_i$ are defined by

$$l_i^r = \frac{\gamma_i}{32\pi^2}\left(\bar{l}_i + \ln\frac{M_{\pi^+}^2}{\mu^2}\right), \qquad k_i^r = \frac{\sigma_i}{32\pi^2}\left(\bar{k}_i + \ln\frac{M_{\pi^+}^2}{\mu^2}\right). \tag{4.8}$$

4.3 Coupling constants F_π and G_π at $\alpha \neq 0$

The quantities F_{π_i} and G_{π_i} are the coupling constants of the isovector axial and pseudoscalar currents to the pion,

$$\langle 0|\, A_\mu^i \,|\pi^k\rangle = ip_\mu\delta^{ik}F_{\pi_i}\,, \qquad \langle 0|\, P^i \,|\pi^k\rangle = \delta^{ik}G_{\pi_i}\,, \tag{4.9}$$

while $\tilde{G}_\pi$ stands for the one-particle matrix element of the isoscalar density,

$$\langle 0|\, P^0 \,|\pi^3\rangle = \tilde{G}_\pi\,. \tag{4.10}$$

The coupling constants F_{π_i}, G_{π_i} and $\tilde{G}_\pi$ are given explicitly in appendix C.

For a constant charge matrix Q the generating functional is invariant under the subgroup of $\mathrm{SU(2)_R} \times \mathrm{SU(2)_L}$ transformations

$$V_\mathrm{R} = \exp\left(ia\tau^3\right), \qquad V_\mathrm{L} = \exp\left(ib\tau^3\right). \tag{4.11}$$

This implies in particular that the relation,

$$F_{\pi^0}M_{\pi^0}^2 = \hat{m}G_{\pi^0} + \frac{m_u-m_d}{2}\tilde{G}_\pi\,, \tag{4.12}$$

holds at $\alpha \neq 0$. We checked that the explicit expressions for $M_{\pi^0}^2, F_{\pi^0}$, G_{π^0} and $\tilde{G}_\pi$ satisfy this relation to order p^4, p^2e^2.

[4] The coefficient of $(m_d-m_u)\,k_7$ differs by a factor $\frac{1}{2}$ from the corresponding term obtained in ref. [7]

5. Extracting the $\pi^+\pi^- \to \pi^0\pi^0$ amplitude from the generating functional

To extract the scattering amplitude from the generating functional, we may concentrate on the pseudoscalar four-point function. The external fields a_μ, v_μ and p^0 are switched off and the scalar field s is set to $\mathcal{M}$. We may even switch $\bar{A}_\mu$ off, because there are no one-photon exchange contributions to $\pi^+\pi^- \to \pi^0\pi^0$ at tree level. Now $Z = Z(p^i, \bar{\phi})$ is a functional of p^i only, depending on p^i both explicitly and implicitly through the equation of motion (2.8).

The pseudoscalar four-point function is given by the coefficient of $p^{i_1}, \dots, p^{i_4}$ in the expansion of $Z(p^i, \bar{\phi})$ in powers of Φ.

5.1 Equation of motion

The solution of the equation of motion (2.8) represents a power series in Φ,

$$\bar{\phi} = \phi^{(1)} + \phi^{(3)} + \cdots . \tag{5.1}$$

To order Φ^4, only the leading order term gives a contribution to the action. Explicitly, $\phi^{(1)}$ reads

$$\phi^{(1)i}(x) = 2BF \int dy \Delta_i(x-y) p^i(y) , \tag{5.2}$$

where Δ_i stands for the Feynman propagator of a pion with mass M_i. Since $\bar{\phi}$ represents an extremum of the action $\int dx \mathcal{L}^{(2)}$, the classical solution may be replaced with the extremum of (4.3),

$$\begin{aligned} \hat{\phi}^i(x) &= Z_i^{\frac{1}{2}} G_{\pi_i} \int dy \Delta_{\pi_i}(x-y) p^i(y) + \mathcal{O}(\Phi^3) , \\ \hat{\phi}(x) &= \bar{\phi}(x) + \mathcal{O}(p^2) , \end{aligned} \tag{5.3}$$

where Δ_{π_i} denotes the Feynman propagator with physical mass M_{π_i}. The shift $\bar{\phi} \to \hat{\phi}$ affects the classical action at order p^6 only,

$$\int dx \mathcal{L}^{(2)}\{\hat{\phi}\} = \int dx \mathcal{L}^{(2)}\{\bar{\phi}\} + \mathcal{O}(p^6) , \tag{5.4}$$

which is beyond one-loop accuracy.

5.2 Recipe

We may determine the scattering matrix element from the on-shell residue of the fourfold pole in the Fourier transform of the pseudoscalar four-point function,

$$\begin{aligned} & i^4 \int dx_1 \dots dx_4 e^{i(p_3x_3 + p_4x_4 - p_1x_1 - p_2x_2)} \langle 0 |\, T P^i(x_1) P^k(x_2) P^l(x_3) P^m(x_4) \,| 0 \rangle = \\ & \quad = \frac{G_{\pi_i} G_{\pi_k} G_{\pi_l} G_{\pi_m}}{(M_{\pi_i}^2 - p_1^2) \dots (M_{\pi_m}^2 - p_4^2)} \langle \pi^l(p_3) \pi^m(p_4)_{\text{out}} \,|\, \pi^i(p_1) \pi^k(p_2)_{\text{in}} \rangle_c + \cdots . \end{aligned} \tag{5.5}$$

The amplitude is related to the connected part of the matrix element through

$$\langle \pi^l(p_3) \pi^m(p_4)_{\text{out}} \,|\, \pi^i(p_1) \pi^k(p_2)_{\text{in}} \rangle_c = i(2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) A^{ik;lm}(s,t,u) . \tag{5.6}$$

Subsequently, we present a simple recipe to extract the $\pi^1\pi^1 \to \pi^3\pi^3$ amplitude from the explicit representation of the generating functional (3.15), (3.18) and (3.19):

- Switch all explicit external fields off, including the field p^i. Now the generating functional $Z = Z(0, \hat{\phi})$ depends on $\hat{\phi}$ only.

- Perform the traces over the flavour matrices σ, $\Gamma_\mu\Gamma_\nu$, $\gamma_\mu\gamma^\mu, \ldots$

- The pion fields are rescaled, according to $\phi^{ri} = Z_i^{-\frac{1}{2}}\hat{\phi}^i$. This step brings the quadratic terms (4.3) to normal form.

- The scattering matrix element may be read off from the terms of order $(\phi^{r1})^2(\phi^{r3})^2$. These contributions generate the on-shell fourfold pole in eq. (5.5).

6. $\pi^+\pi^- \to \pi^0\pi^0$ scattering amplitude

As a check, we calculated the $\pi^+\pi^- \to \pi^0\pi^0$ amplitude to $\mathcal{O}(p^4, e^2p^2)$ from (3.15), (3.18) and (3.19). We compared the thus obtained expression with the result of Knecht and Urech [7], who used a photon mass as an infrared regulator. Upon identifying the infrared pole term in dimensional regularization with

$$\lambda_{IR} = -\frac{1}{32\pi^2}\left(1 + \ln\frac{m_\gamma^2}{\mu^2}\right), \tag{6.1}$$

the two results are in agreement.

6.1 Infrared singularities

The on-shell matrix element contains infrared singularities generated by virtual photon contributions. Since we used dimensional regularization to treat the infrared pole terms, the amplitude is well defined in $d \neq 4$ only. The essential loop function, which develops an infrared singularity as $d \to 4$, is the scalar triangle diagram $G_d(s)$ with one photon and two charged on-shell pion propagators. A discussion of the analytic properties of $G_d(s)$, as well as an explicit representation of this function, is given in appendix B.

Expanding the amplitude around threshold $s = 4M_{\pi^+}^2$ leads to

$$\mathrm{Re}A^{+-00} = -\frac{e^2}{16F^2}\frac{M_{\pi^+}}{|\mathbf{q}|}\left(3M_{\pi^+}^2 + \Delta_\pi\right) + \mathrm{Re}A_{\mathrm{thr}}^{+-00} + \mathcal{O}(|\mathbf{q}|), \tag{6.2}$$

in the Condon-Shortley phase conventions. Here, $\mathbf{q}$ denotes the relative 3-momentum of the charged pions and the Coulomb pole stems from the infrared region of the scalar triangle diagram (B.5). The quantity $\mathrm{Re}A_{\mathrm{thr}}^{+-00}$ is free of infrared singularities.

6.2 Amplitude at threshold

We may now expand $\mathrm{Re}A_{\mathrm{thr}}^{+-00}$ in powers of the isospin breaking parameter δ,

$$\alpha \doteq \frac{e^2}{4\pi} = \mathcal{O}(\delta), \qquad (m_d - m_u)^2 = \mathcal{O}(\delta), \tag{6.3}$$

according to

$$-\frac{3}{32\pi}\mathrm{Re}A_{\mathrm{thr}}^{+-00} = a_0 - a_2 + h_1(m_d - m_u)^2 + h_2\alpha + \mathcal{O}\Big(\delta^{3/2}\Big). \tag{6.4}$$

The combination $a_0 - a_2$ denotes the S-wave scattering length difference in the isospin symmetry limit $m_u = m_d$, $\alpha = 0$ [2],

$$a_0 - a_2 = \frac{9M_{\pi^+}^2}{32\pi F^2}\left[1 + \frac{M_{\pi^+}^2}{288\pi^2 F^2}\left(33 + 8\bar{l}_1 + 16\bar{l}_2 - 3\bar{l}_3\right)\right] + \mathcal{O}\big(\hat{m}^3\big). \tag{6.5}$$

The coefficients h_1 and h_2 of the isospin breaking contributions are given by [22, 23],

$$\begin{aligned} h_1 &= \mathcal{O}(\hat{m})\,, \\ h_2 &= \frac{3\Delta_\pi^{e.m.}}{32\pi\alpha F^2}\left[1 + \frac{M_{\pi^+}^2}{12\pi^2F^2}\left(\frac{23}{8} + \bar{l}_1 + \frac{3}{4}\bar{l}_3\right)\right] + \frac{3M_{\pi^+}^2}{256\pi^2F^2}p(k_i) + \mathcal{O}\big(\hat{m}^2\big), \end{aligned} \tag{6.6}$$

where $\Delta_\pi^{e.m.}$ denotes the physical mass difference at $m_u = m_d$,

$$\Delta_\pi^{e.m.} = \Delta_\pi\big|_{m_u=m_d}\,, \tag{6.7}$$

and $p(k_i)$ stands for the following combination of low-energy constants

$$p(k_i) = -30 + 9\bar{k}_1 + 6\bar{k}_3 + 2\bar{k}_6 + \bar{k}_8 + \frac{4Z}{3}\left(\bar{k}_1 + 2\bar{k}_2 + 6\bar{k}_4 + 12\bar{k}_6 - 6\bar{k}_8\right). \tag{6.8}$$

7. Summary and outlook

In the present paper we constructed an explicit representation of the generating functional for $\mathrm{SU}(2)_\mathrm{R} \times \mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{V}$ with virtual photons at first non-leading order in the low-energy expansion $[\mathcal{O}(p^4, e^2p^2)]$. The explicit form includes graphs, where up to two pions and at most one photon are running in the loop.

It has been shown in [7] that the ultraviolet divergences of the one-loop functional are absorbed by the corresponding counterterms of $\mathcal{O}(p^4, e^2p^2)$. For non-exceptional 4-momenta the generating functional is thus finite as $d \to 4$. Nevertheless, if we want to extract on-shell matrix elements, we encounter in general infrared singularities, generated by soft photon contributions. We use dimensional regularization to treat these infrared pole terms and therefore specify the explicit form of the generating functional in $d \neq 4$ dimensions.

As a check of the one-loop functional, we evaluated the $\pi^+\pi^- \to \pi^0\pi^0$ scattering amplitude at order p^4, e^2p^2. The thus obtained result agrees with the one in ref. [7].

Furthermore, we specify the electromagnetic corrections to the decay constants F_π and G_π, which measure the one-particle matrix elements of the isovector axial and pseudoscalar currents.

The extension to $\mathrm{SU}(3)_\mathrm{R} \times \mathrm{SU}(3)_\mathrm{L}$ is in progress [19]. It will provide a comparison with the $\pi - K$ amplitudes already available in the literature [18].

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A. Kernels

Here, we explain the notation used in subsection 3.3. We consider the Fourier transforms ($s = p^2$)

$$\begin{aligned} K_\mu\left(p, M_i^2, M_k^2\right) &= \int d^d z e^{ipz} K_\mu\left(z, M_i^2, M_k^2\right), \\ M_{\mu\nu}^r\left(p, M_i^2, M_k^2\right) &= \int d^d z e^{ipz} M_{\mu\nu}^r\left(z, M_i^2, M_k^2\right). \end{aligned} \tag{A.1}$$

The kernel K_μ remains ultraviolet finite as $d \to 4$,

$$K_\mu\left(p, M_i^2, M_k^2\right) = \frac{ip_\mu}{2}\frac{M_i^2 - M_k^2}{s}\bar{J}\left(s, M_i^2, M_k^2\right), \tag{A.2}$$

where $\bar{J}(s) = J(s) - J(0)$ and

$$J\left(s, M_i^2, M_k^2\right) = \frac{1}{i}\int \frac{d^d q}{(2\pi)^d}\frac{1}{(M_i^2 - q^2)(M_k^2 - (p-q)^2)}. \tag{A.3}$$

The renormalized kernel $M_{\mu\nu}^r$ reads:

$$M_{\mu\nu}^r\left(p, M_i^2, M_k^2\right) = -(p_\mu p_\nu - g_{\mu\nu} s) M^r\left(s, M_i^2, M_k^2\right) - g_{\mu\nu} L\left(s, M_i^2, M_k^2\right), \tag{A.4}$$

with

$$\begin{aligned} M^r(s, M_i^2, M_k^2) &= \frac{1}{4(d-1)s} \times \\ &\times\left\{\left[s - 2\left(M_i^2 + M_k^2\right)\right]\bar{J}\left(s, M_i^2, M_k^2\right) + \frac{d}{s}\left(M_i^2 - M_k^2\right)^2 \bar{\bar{J}}\left(s, M_i^2, M_k^2\right)\right\} - \\ &-\frac{1}{6}k\left(M_i^2, M_k^2\right) + \frac{1}{288\pi^2}, \\ L(s, M_i^2, M_k^2) &= \frac{1}{4s}(M_i^2 - M_k^2)^2 \bar{J}\left(s, M_i^2, M_k^2\right), \end{aligned} \tag{A.5}$$

and

$$\begin{aligned} \bar{\bar{J}}(s) &= \bar{J}(s) - s\bar{J}'(0), \\ k\left(M_i^2, M_k^2\right) &= \frac{1}{32\pi^2(M_i^2 - M_k^2)}\left[M_i^2 \ln\frac{M_i^2}{\mu^2} - M_k^2 \ln\frac{M_k^2}{\mu^2}\right]. \end{aligned} \tag{A.6}$$

We list the ultraviolet finite kernels K_μ and $M_{\mu\nu}^r$ in $d \neq 4$ dimensions, because we encounter an infrared singularity, if we expand the charged pion self-energy around its physical mass.

This pole term, which comes from $\bar{J}'(M_{\pi^+}^2, M_{\pi^+}^2, 0)$, leads to an infrared divergent scaling factor Z_+ in eq. (4.4).

Furthermore, we introduce the Fourier transform

$$T(p_1, p_2) = \int d^d u \, d^d v \, e^{ip_1 u} e^{ip_2 v} T(u, v) \,, \tag{A.7}$$

and similarly for the other kernels $T_{i\mu}, T_{i\mu\nu}$ and $T^{\mu}_{\mu\nu}$ in eq. (3.19). Then we have

$$\begin{aligned}
T(p_1, p_2) &= I \,, \\
T_{1\mu}(p_1, p_2) &= ip_{1\mu} I - 2iI_\mu \,, \\
T_{2\mu}(p_1, p_2) &= ip_{2\mu} I - 2iI_\mu \,, \\
T_{3\mu}(p_1, p_2) &= i(p_1 + p_2)_\mu I - 2iI_\mu \,, \\
T_{1\mu\nu}(p_1, p_2) &= 4I_{\mu\nu} - 2(p_1 + p_2)_\mu I_\nu - 2p_{1\nu} I_\mu + (p_1 + p_2)_\mu p_{1\nu} I \,, \\
T_{2\mu\nu}(p_1, p_2) &= 4I_{\mu\nu} - 2(p_1 + p_2)_\nu I_\mu - 2p_{2\mu} I_\nu + p_{2\mu}(p_1 + p_2)_\nu I \,, \\
T_{3\mu\nu}(p_1, p_2) &= 4I_{\mu\nu} - 2p_{1\mu} I_\nu - 2p_{2\nu} I_\mu + p_{1\mu} p_{2\nu} I \,, \\
T^{\mu}_{\mu\nu}(p_1, p_2) &= i\Big\{ - 8I^{\mu}_{\mu\nu} + 4(p_1 + p_2)^\mu I_{\mu\nu} + 4(p_1 + p_2)_\nu I^{\mu}_{\mu} - 2(p_1 \cdot p_2) I_\nu - \\
&\qquad - 2(p_1 + p_2)_\mu (p_1 + p_2)_\nu I^\mu + (p_1 \cdot p_2)(p_1 + p_2)_\nu I \Big\} \,,
\end{aligned} \tag{A.8}$$

with

$$\{I, I_\mu, I_{\mu\nu}, I_{\mu\nu\rho}\} = \frac{1}{i} \int \frac{d^d k}{(2\pi)^d} \frac{\{1, k_\mu, k_\mu k_\nu, k_\mu k_\nu k_\rho\}}{(M_i^2 - (p_1 - k)^2)(M_k^2 - (p_2 - k)^2)(-k^2)} \,. \tag{A.9}$$

The integrals I and I_μ are ultraviolet finite, but I develops an infrared singularity for on-shell 4-momenta (see appendix B). The functions $I_{\mu\nu}$ and $I_{\mu\nu\rho}$ have ultraviolet poles at $d = 4$:

$$\begin{aligned}
T_{i\mu\nu}(p_1, p_2) &= T^r_{i\mu\nu}(p_1, p_2) + 2g_{\mu\nu}\lambda \qquad i = 1, \ldots, 3 \,, \\
T^{\mu}_{\mu\nu}(p_1, p_2) &= T^{\mu r}_{\mu\nu}(p_1, p_2) + \frac{2}{3} i\lambda(d - 1)(p_1 + p_2)_\nu \,.
\end{aligned} \tag{A.10}$$

For non-exceptional 4-momenta the renormalized quantities $T^r_{i\mu\nu}$ and $T^{\mu r}_{\mu\nu}$ remain finite as $d \to 4$.

B. Scalar vertex function

The scalar vertex function $G_d(s)$ is given by

$$G_d(s) = \frac{1}{i} \int \frac{d^d k}{(2\pi)^d} \frac{1}{M_{\pi^+}^2 - (p_1 - k)^2} \frac{1}{M_{\pi^+}^2 - (p_2 + k)^2} \frac{1}{-k^2} \,, \tag{B.1}$$

where p_1, p_2 denote the on-shell 4-momenta of the incoming π^+, π^- mesons and $s = (p_1 + p_2)^2$. The vertex function is ultraviolet finite, but develops an infrared singularity at $d = 4$. (At $p_i^2 \neq M_{\pi^+}^2$, the singularity would be absent).

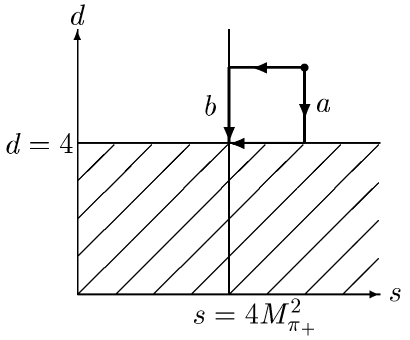

Figure 2: Two different paths to approach to threshold. The shaded region is excluded since the integral G_d does converge for $d > 4$ only.

For $s \geq 4M_{\pi^+}^2$, the function has a cut along the positive real axis. Expanding the vertex function in $d-4$ for arbitrary $s \geq 4M_{\pi^+}^2$ yields

$$G_d(s) = \frac{1}{32\pi^2 s\sigma}\left\{2\pi^2 + 4\mathrm{Li}\frac{1+\sigma}{1-\sigma} + \left(\ln\frac{1+\sigma}{1-\sigma}\right)^2 - 2\ln\frac{1+\sigma}{1-\sigma}\left(1 + 32\pi^2\lambda_{IR} + \ln\frac{M_{\pi^+}^2}{\mu^2}\right)\right\} + \\ + \frac{i}{16\pi s\sigma}\left(1 + 32\pi^2\lambda_{IR} + \ln\frac{s\sigma^2}{\mu^2}\right) + \mathcal{O}(d-4)\,, \tag{B.2}$$

with

$$\sigma = \sqrt{1 - \frac{4M_{\pi^+}^2}{s}} \tag{B.3}$$

and

$$\mathrm{Li}(z) \doteq \mathrm{Li}_2(1-z) = \int_1^z du \frac{\ln u}{1-u}\,. \tag{B.4}$$

If we expand (B.2) near threshold $s = 4M_{\pi^+}^2 + 4\mathbf{q}^2$, which corresponds to path a in figure 2, we find

$$G_d(s) = \frac{1}{64M_{\pi^+}^2}\left\{\frac{M_{\pi^+}}{|\mathbf{q}|} - \frac{2}{\pi^2}\left(3 + 32\pi^2\lambda_{IR} + \ln\frac{M_{\pi^+}^2}{\mu^2}\right) - \frac{|\mathbf{q}|}{2M_{\pi^+}} + \right. \\ \left. + \frac{i}{\pi}\left(\frac{M_{\pi^+}}{|\mathbf{q}|} - \frac{|\mathbf{q}|}{2M_{\pi^+}}\right)\left(1 + 32\pi^2\lambda_{IR} + \ln\frac{4\mathbf{q}^2}{\mu^2}\right)\right\} + \mathcal{O}(d-4) + \mathcal{O}(\mathbf{q}^2)\,, \tag{B.5}$$

where $\mathbf{q}$ denotes the relative momentum of the incoming pions in the CM frame. The singular contributions $\sim 1/|\mathbf{q}|$ are generated by small values of the variable of integration k in (B.1).

On the contrary, if we first evaluate the vertex function at threshold in $d \neq 4$ dimensions

$$G_d(4M_{\pi^+}^2) = \frac{C_d}{M_{\pi^+}^2}\frac{1}{(d-4)(d-5)}M_{\pi^+}^{d-4}\,, \qquad C_d = \frac{1}{(4\pi)^{\frac{d}{2}}}\Gamma\left(3 - \frac{d}{2}\right) \tag{B.6}$$

and then expand in $d-4$, i.e we choose path b in figure 2

$$G_d(4M_{\pi^+}^2) = -\frac{1}{M_{\pi^+}^2}\left[\lambda_{IR} + \frac{1}{32\pi^2}\left(3 + \ln\frac{M_{\pi^+}^2}{\mu^2}\right)\right] + \mathcal{O}(d-4)\,, \tag{B.7}$$

we are left with the constant term in eq. (B.5).

C. Coupling constants F_π and G_π

In presence of electromagnetic interactions the coupling constants of the isovector axial and pseudoscalar currents to the pion are given by,[5]

$$\begin{aligned} F_{\pi^+} &= F_\pi + F\Bigg\{2e^2\left[\lambda_{IR} + \frac{1}{32\pi^2}\left(\ln\frac{M^2_{\pi^+}}{\mu^2} - 2\right)\right] - \frac{e^2 Z}{8\pi^2}\bar{l}_4 - \frac{e^2}{64\pi^2}\times \\ &\qquad \times\left[\left(3+\frac{4Z}{9}\right)\bar{k}_1 - \frac{40Z}{9}\bar{k}_2 - \bar{k}_9\right] + \frac{M^2_{\pi^+} - 2Ze^2F^2}{32\pi^2F^2}\ln\frac{M^2_{\pi^+}}{M^2_{\pi^0}} + \mathcal{O}\big(e^2 m_q\big)\Bigg\}, \\ F_{\pi^0} &= F_\pi - F\left\{\frac{e^2 Z}{8\pi^2}\bar{l}_4 + \frac{e^2}{64\pi^2}\left[\left(3+\frac{4Z}{9}\right)\bar{k}_1 - \frac{40Z}{9}\bar{k}_2 - 3\bar{k}_3 - 4Z\bar{k}_4\right] + \mathcal{O}(e^2 m_q)\right\} \end{aligned} \tag{C.1}$$

and

$$\begin{aligned} G_{\pi^+} &= G_\pi + 2BF\Bigg\{2e^2\left(\lambda_{IR} + \frac{1}{32\pi^2}\ln\frac{M^2_{\pi^+}}{\mu^2}\right) + \frac{e^2 Z}{16\pi^2}\bar{l}_3 + \frac{e^2}{64\pi^2}\times \\ &\qquad \times\Bigg[\left(3+\frac{4Z}{9}\right)\bar{k}_1 - \frac{40Z}{9}\bar{k}_2 - \frac{2}{9}(5+4Z)\bar{k}_5 + \frac{10}{9}(1+8Z)\bar{k}_6 + \\ &\qquad + (1-8Z)\bar{k}_8\Bigg] + \frac{4e^2}{9}k_7 + \mathcal{O}(e^2 m_q)\Bigg\}, \\ G_{\pi^0} &= G_\pi + 2BF\Bigg\{\frac{e^2 Z}{16\pi^2}\left(\bar{l}_3 - 2\bar{l}_4\right) + \frac{e^2}{64\pi^2}\Bigg[\left(3+\frac{4Z}{9}\right)\bar{k}_1 - \frac{40Z}{9}\bar{k}_2 - 3\bar{k}_3 - 4Z\bar{k}_4 - \\ &\qquad - \frac{2}{9}(5+4Z)\bar{k}_5 + \frac{10}{9}(1+8Z)\bar{k}_6\Bigg] + \frac{4e^2}{9}k_7 + \\ &\qquad + \frac{M^2_{\pi^+} - 2e^2F^2Z}{32\pi^2F^2}\ln\frac{M^2_{\pi^+}}{M^2_{\pi^0}} + \mathcal{O}(e^2 m_q)\Bigg\}. \end{aligned} \tag{C.2}$$

The quantities F_π and G_π denote the coupling constants at $\alpha = 0$ and $m_u = m_d$ [2],

$$\begin{aligned} F_\pi &= F\left[1 + \frac{M^2_{\pi^+}}{16\pi^2F^2}\bar{l}_4 + \mathcal{O}\big(m_q^2\big)\right], \\ G_\pi &= 2BF\left[1 - \frac{M^2_{\pi^+}}{32\pi^2F^2}\left(\bar{l}_3 - 2\bar{l}_4\right) + \mathcal{O}\big(m_q^2\big)\right]. \end{aligned} \tag{C.3}$$

Due to the infrared pole term, F_{π^+} and G_{π^+} are well defined in $d \neq 4$ dimensions only.

The coupling constant of the isoscalar P^0 to the pion is of $\mathcal{O}((m_d - m_u), e^2)$

$$\tilde{G}_\pi = \frac{4B^2(m_d - m_u)}{F}l_7 + \frac{8BFe^2}{3}k_7\,. \tag{C.4}$$

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